De Bruijn and Kautz Bus Networks with Implementation via Hologram Waveguides

Jinghuai Fa and Ray T. Chen
Microelectronics Research Center
Department of Electrical and Computer Engineering
University of Texas at Austin, Texas 78712-1084

ABSTRACT

In this paper, we discuss multihop optical bus interconnection networks. For massive parallel computing systems, such interconnection can interconnect large number of processing elements and reduce system latency at the same time. We give the qualitative results of de Bruijn bus network and Kautz bus network as an engineering options based on combinatorial hypergraph theory and a uniform alphabetic constructive method. We also give an implementation method via polymer based hologram waveguide.


1. INTRODUCTION

Interconnection network design problems become more and more important nowadays both in massive parallel computing systems and VLSI systems. When nodes in the system to be connected become large, this problem could be very hard in order to enhance the interconnect performance in a low cost. The basic styles of interconnection are point to point interconnection and broadcast interconnection. In point to point communication, multihops may needed to fully interconnect all the nodes in the system. In the broadcast interconnection, a simple way is attach the processors all to a single bus. Between them, there is the third method -- multicast interconnection -- the grouped interconnection which communicate in point to point style between groups and in broadcast style within groups. One can image multicast as interconnected bus system.

There are many important parameters in designing interconnect network. Their importance varies according to different implementation techniques. We list here some of the major parameters in network.

1) Diameter D: the maximum distance between any two nodes in the system. It represents maximum hops or times a message needed in order to go through any two nodes in system, and characterize the communication delay of a system. We expect a small diameter.

2) Degree Δ: the number of transmitter-receiver pairs each node has. The more the transmitters a node has, the easier it can be connected to outside world, but the more cost.

3) Volume v: the maximum number of nodes a bus can accommodate.
4) System size $n$: the total number of nodes a network can interconnect.

Although there are many other parameters for a network, we focus in this paper on the parameters mentioned above, discuss the relations between those parameters, and network design issue based on the parameters. We restrict our discussion in regular network, that is, each node has the same number of $\Delta$.

We hope a big $n$ network with small diameter $D$, or small degree $\Delta$, or small bus volume $r$. But these parameters are contradictory. For instance, with fixed degree $\Delta$, we need more hops (larger $D$) or larger bus volume to reach more nodes. With fixed diameter $D$(fixed hops), we need more transmitter-receiver pairs $\Delta$ or larger bus volume $r$ to reach more nodes. There are many options according to different network, but the maximum number of nodes is bounded by Moore bound:

$$n(\Delta, D, r) \leq 1 + \Delta(r - 1) \sum_{i=0}^{D-1} (\Delta - 1)^i (r - 1)^i$$ for undirected interconnects,
$$n(\Delta, D, r) \leq \prod_{i=0}^{D-1} (\Delta r)^i$$ for directed interconnects.

An interconnection network attain this Moore bound is known as Moore geometry, and it is almost impossible to attain that bound. There are many works done to try to search graphs in order to approach the Moore bound as near as possible. Some results are summarized in [4] [13]. Among them, de Bruijn network and Kautz network have better properties for system interconnection. They have been used in WDM, VLSI and computer systems interconnections.

This paper is organized as follows. Section 2 discuss point to point interconnection network implemented using de Bruijn network and Kautz network. Section 3 discuss bus interconnection network implemented using de Bruijn and Kautz graphs. Section 4 discuss more general cases by using hypergraph technology. And Section 5 comes with conclusion.

2. DE BRUIJN NETWORK AND KAUTZ NETWORK

In this section, we discuss point to point interconnection implemented using de Bruijn network and Kautz network. No bus is considered here. Before discussing de Bruijn and Kautz network, we will first take a look at $k$-cube(or hypercube) which is the most popular and currently in used point to point interconnection network. We can describe $k$-cube as nodes represented by binary sequences of length $k$(k tuple). Any two nodes are interconnected if and only if the two binary sequences have one bit difference. Such, a $k$-cube has diameter $k$ with $2^k$ nodes each of which has degree $k$. Using standard terminology here, $D=k$, $\Delta=k$, and by Moore bound,

$$n(\Delta, D) \leq 1 + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1} = 1 + k(k - 1) + \cdots + k(k - 1)^{k-1}$$

$$= \frac{k(k - 1)^k - 2}{k - 2}$$

We can see from above that the number of nodes can interconnected by $k$-cube( that is $2^k$ ) are far from Moore bound when $k$ is large ($>4$) which means there may exist some
better networks. In fact, de Bruijn network and Kautz network are such networks. We note that, if two networks \( a, b \), with the same \( \Delta \) and \( D \), can interconnect \( n_a \) nodes and \( n_b \) nodes respectively, and \( n_a \gg n_b \), then we can conclude that, if we interconnect \( n_b \) nodes using network \( a \), then it need only smaller number of average hops than using \( b \) to interconnect \( n_b \) nodes which makes system faster. Or network may need less degree for each node which means saving of transmitter-receiver pairs. Following table shows the comparison of nodes can be connected using different networks for some \( \Delta \) and \( D \), and their Moore bound.

<table>
<thead>
<tr>
<th>( n(\Delta, D) )</th>
<th>( \Delta = D = 4 )</th>
<th>( \Delta = D = 6 )</th>
<th>( \Delta = D = 8 )</th>
<th>( \Delta = D = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>k-cube</strong></td>
<td>16</td>
<td>64</td>
<td>256</td>
<td>1024</td>
</tr>
<tr>
<td><strong>de Bruijn</strong></td>
<td>16</td>
<td>729</td>
<td>65536</td>
<td>9765625</td>
</tr>
<tr>
<td><strong>Kautz</strong></td>
<td>24</td>
<td>972</td>
<td>81920</td>
<td>11718750</td>
</tr>
<tr>
<td><strong>Moore bound</strong></td>
<td>161</td>
<td>23437</td>
<td>7.7E+6</td>
<td>4.36E+9</td>
</tr>
</tbody>
</table>

Table 1 Number of nodes can be connected using different networks. Here de Bruijn and Kautz are undirected (see last part of this section)

de Bruijn network

There are several ways to define de Bruijn network. For example, words on an alphabet, line digraph iteration, integer congruence. Here we use words on alphabet.

The de Bruijn network \( B(\Delta, D) \) is a digraph (direct graph) that each vertex is a word of length \( D \) on an alphabet \( \Sigma \) of \( \Delta \) letters. Two nodes \( u \) and \( v \) (that is two words) will have an edge from \( u \) to \( v \) (denoted by \( u \rightarrow v \)) iff \( v \) is a left shifted version of \( u \), that is, if \( u = u_1u_2 \ldots u_D \), then \( u \rightarrow v \) exists iff \( v = u_2u_3 \ldots u_D\alpha \), for any \( \alpha \in \Sigma \). Such, each node in de Bruijn network has out-degree \( \Delta \) and in-degree \( \Delta \). Any two nodes in de Bruijn network need at most \( D \) hops to reach each other. We can show this easily as follows. Suppose \( u = u_1u_2 \ldots u_D \), \( v = v_1v_2 \ldots v_D \), then, \( u_1u_2 \ldots u_D \rightarrow u_2u_3 \ldots u_Dv_1, \rightarrow u_3 \ldots u_Dv_1v_2 \rightarrow \ldots \rightarrow v_1v_2 \ldots v_D \), has \( D \) hops. Fig 1 shows a \( B(2, 3) \) network.

![Fig. 1 De Bruijn network B(2, 3)](image)

For a word of length \( D \) on an alphabet \( \Sigma \) having \( \Delta \) elements, on each position in the word, we have \( \Delta \) options, so totally we have \( \Delta^D \) different words. Such, de Bruijn
network \( B(\Delta, D) \) has \( \Delta^D \) nodes. And since de Bruijn network is regular graph, each node has \( \Delta \) out-going edges, it has totally \( e = n \times \Delta = \Delta^{D+1} \) edges.

**Kautz network**

The Kautz network \( K(\Delta, D) \) is a digraph that each vertex is a word of length \( D \) on an alphabet \( \Sigma \) of \( \Delta+1 \) letters, with restriction that any two consecutive symbols in a word are different. Two nodes \( u \) and \( v \) (that is two words) will have an edge from \( u \) to \( v \) (denoted by \( u \rightarrow v \)) iff \( v \) is a left shifted version of \( u \), that is, if \( u = u_1u_2\cdots u_D \), then \( u \rightarrow v \) exists iff \( v = u_2u_3\cdots u_Du_1 \), for any \( \alpha \neq u_D, \alpha \in \Sigma \). Such, each node in Kautz network has out-degree \( \Delta \) and in-degree \( \Delta \). Any two nodes in Kautz network need at most \( D \) hops to reach each other. We can show this easily as follows. Suppose \( u = u_1u_2\cdots u_D, v = v_1v_2\cdots v_D \), then, if \( v_1 \neq u_D \), \( u_1u_2\cdots u_D \rightarrow u_2u_3\cdots u_Dv_1 \rightarrow \cdots \rightarrow u_Dv_2\cdots v_D \), it has \( D \) hops. If \( v_1 = u_D \), then \( u_1u_2\cdots u_D \rightarrow u_2\cdots u_Dv_1 \rightarrow \cdots \rightarrow u_Dv_2\cdots v_D = v_1v_2\cdots v_D \). Fig 2 shows a \( K(2, 2) \) and a \( K(2, 3) \) network.

![Kautz Networks](image)

**Fig. 2** Kautz Networks. Left: \( K(2, 2) \). Right: \( K(2, 3) \).

For a word of length \( D \) on an alphabet \( \Sigma \) having \( \Delta+1 \) elements, on the first position of the word we have \( \Delta+1 \) options, and on each of the rest positions in the word, we have \( \Delta \) options, so totally we have \( (\Delta+1)^{\Delta-1} = \Delta^D + \Delta^{D-1} \) different words. Such Kautz network \( K(\Delta, D) \) has \( \Delta^D + \Delta^{D-1} \) nodes. And since Kautz network is regular graph, each node has \( \Delta \) out-going edges, it has totally \( e = n \times \Delta = \Delta^{D+1} + \Delta^D \) edges.

**Undirected de Bruijn and Kautz networks**

Undirected network is obtained from the directed one simply by replacing the directed edges with undirected edges, and remove possible loops and parallel edges. We denote undirected de Bruijn network with edge degree \( \Delta \) and diameter \( D \) as \( UB(\Delta, D) \), and undirected Kautz network \( UK(\Delta, D) \). They can connect \( (\Delta/2)^D + (\Delta/2)^{D-1} \) nodes for \( UK(\Delta, D) \); \( (\Delta/2)^D \) nodes for \( UB(\Delta, D) \).
3. BUS NETWORK DESIGN WITH DE BRUIJN NETWORK

With the nice property of de Bruijn network, we will replace each edge of UB(Δ, D) with a bus. Such bus system has maximum diameter D+1. UB(2,3) and UK(2,2) are shown as follows, where each edge is replaced by a bus.

![Diagram of UB(2,3) and UK(2,2) with buses replacing edges.]

In fig. 3, each white dot represents a fan-in/fan-out point where signal can send to (or receive from) other fan-in/fan-out point on the same bus, including the end numbered end points which are also fan-in/fan-out points. For example, any white dot on the bus between 01 and 20 can reach 02 in two hops; any white dot on the bus between 000 and 100 can reach 111 in three hops. Here, each bus is a polymer based hologram waveguide.

Hologram waveguide is a optical device. Laser beam inject into hologram waveguide and will be diffracted at hologram grating layer according to Bragg diffraction condition. Fig. 4 shows the diffraction diagram for fan-in. On the left, \( |K_\text{in}^\perp| = 2\pi n / \lambda \), where \( n \) is the refractive index, \( \lambda \) is the wavelength of the laser beam. \( |K_\text{out}^\perp| \) is the resulting beam vector. \( K_1 \) is the needed grating vector to get the resulting vector, which is perpendicular to the grating orientation. \( |K_1| = 2\pi / \Lambda \), where \( \Lambda \) is the period of the grating. Our objective is to make hologram having the pattern of \( K_1 \) (properly chose \( \Lambda \) and the direction of \( K_1 \)) to redirect the laser beam to the left in 45 degree. The right side of Fig. 4 shows another grating pattern (\( K_2 \), perpendicular to the grating orientation) which redirects the laser beam to the right in 45 degree. \( K_{\text{undi}} \) represents some part of beam that is not diffracted.

![Diagram of diffraction at hologram grating layer.]
Fig. 4  Diffraction diagram for fan-in

Fig. 5 shows diffraction diagram for fan-out. Since our design is based on 45 degree principle, surface-normal fan-in results in several fan-out that also are surface-normal according to properly grating $K_1$ and $K_2$ as shown in the vector diagram in fig. 5.

![Diagram of diffraction for fan-in](image)

Fig. 5  Diffraction diagram for fan-out

Fig. 6 shows a multiplexed hologram grating. Two exposures applied to one light-sensitive polymer attached to substrate. When one processor sends a light beam signal, any other processor receives the fan-out signal. Multiplexed vector diagrams are also shown in fig. 6.

![Diagram of multiplexed polymer based hologram](image)

Fig. 6  Multiplexed polymer based hologram waveguide.

Fig. 7 shows our setup for recording the hologram. The laser beam from an Argon laser is splitted into two beam via beam splitter. One is the objective beam, another is the reference beam. Two beam If we denote the angle between these two beam as $\theta$, then the grating period $\Lambda$ will be $\Lambda = \lambda / (2n_r \sin(\theta/2))$, where $\lambda$ is the wavelength and $n_r$ is the refractive index of the air.
The reconstruction wavelengths may differ from the recording wavelength. Such, to form a slanted grating coupler which converts a vertical incident wave into a total internal reflection substrate guided mode with diffraction angle $\alpha$ (there is 45 degree), the two incident angles of the recording beams are[22]

\[
\theta_1 = \sin^{-1}\left\{ \frac{n}{n_r} \sin \left[ \frac{\alpha}{2} \pm \sin^{-1}\left( \frac{\lambda_b}{\lambda_r} \sin \frac{\alpha}{2} \right) \right] \right\},
\]

\[
\theta_2 = \sin^{-1}\left\{ \frac{n}{n_r} \sin \left[ \frac{\alpha}{2} \pm \sin^{-1}\left( \frac{\lambda_b}{\lambda_r} \sin \frac{\alpha}{2} \right) \right] \right\},
\]

where $n$ is the reflective index of the hologram, $\lambda_b$ and $\lambda_r$ represent the wavelengths of the recording and the reconstruction waves, respectively.

On the right of fig. 7, there shows the resulting multiplexed hologram waveguide bus. By replacing each edge of UB(2,3) with such bus, we get Fig. 8.
If maximal \( r \) devices can be attached to each bus, then, for de Bruijn \( B(\Delta,D) \), total number of devices is \( \Delta^D (r\Delta - 2\Delta + 1) \). And for Kautz \( K(\Delta,D) \), total number of devices will be \( (\Delta^D + \Delta^{D-1})(r\Delta - 2\Delta + 1) \).

Note that, the buses here are bidirectional. If they are unidirectional, the results hold.

In the next section, we will see that bus network can accommodate more devices with fixed parameters of \( \Delta, D, r \) by using hypergraph.

4. BUS NETWORK DESIGN WITH HYPERGRAPH

Hypergraph

A graph is a set of nodes connected by edges. Each edge connected only two nodes. Hypergraph extend a graph in that each edge can connect more than two nodes, such edge called hyperedge( for more detail, see[8]). A \( n \) nodes bus system is a hypergraph with \( n \) nodes and one hyperedge. A hypergraph is denoted as \( (\Delta,D,r) \), where \( r \) is the volume of the hyperedge( maximum nodes can be accommodated within an hyperedge), \( \Delta \), and \( D \) as defined before. The maximum number of nodes a network may has is bounded by Moore bound:

\[
n(\Delta,D,r) \leq 1 + \Delta(r-1)\sum_{i=0}^{D-1} (\Delta-1)^i (r-1)^i.
\]

Dual Hypergraph

Given a hypergraph \( H \), we can take its hyperedge as another hypergraph’s node( the volume \( r \) of the hyperedge becomes the degree \( \Delta \) of nodes in the new hypergraph), its node as the hyperedge of the new hypergraph( node degree \( \Delta \) becomes the bus volume \( r \) of the new hypergraph. We will call this new hypergraph the dual hypergraph of original one, denoted as \( H^* \). It is well known [8] that, if \( H \) is a \( (\Delta,D,r) \)-hypergraph, then its dual hypergraph is a \( (r, D^*, \Delta) \)-hypergraph, where \( D-1 \leq D^* \leq D+1 \). This means the difference of diameters of a hypergraph and its dual is at most one. It’s interesting to construct hypergraph \( (2, D, \Delta) \) from the ordinary graph \( (\Delta, D, 2) \) ( that is the regular graph \( (\Delta, D) \)). Such, from ordinary de Bruijn network( or Kautz network ) with node degree \( \Delta \) and diameter \( D \) ( note that regular graph has bus volume \( r=2 \), by using hypergraph dual operation, we get a hypergraph with bus volume \( \Delta \) and each node on two buses.

De Bruijn Bus Network and Kautz Bus Network

De Bruijn Bus Network

Let’s denote de Bruijn bus network as \( B(\Delta, D, k) \) where \( \Delta \) is the degree of each node, \( D \) is the maximum hop between any two nodes in the network, \( k \) is the number of nodes can be connected to a bus. We will discuss two different cases. First, we will discuss the case where each node’s transmitter and receiver may attach to different buses, each bus can accommodate \( k \) transmitters and \( k \) receivers from different nodes, as denoted as \( B(\Delta, D, k) \). In the second case, we will discuss the the case where the
transmitter and receiver of each nodes are unseparable, the maximum number of nodes a bus can accommodate is $k$, we denoted such de Bruijn network as $UB(\Delta, D, k)$.

Since the representation of bus network can be done using graph, we need hypergraph representation where each edge is a hyperedge which means there may be more than 2 nodes on a edge. Each hyperedge is a bus. If there are $k$ nodes are incident to the hyperedge, then it means there are $k$ transmitters of different nodes are attached to the bus. If there are $k$ nodes are incident from the hyperedge, then it means there are $k$ receivers attached to the bus.

De Bruijn bus network $B(\Delta, D, k)$ are constructed as follows. We code each node as an element $w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$, where $w_i \in \Sigma_k$, $\Sigma_k$ is an alphabet with $k$ elements, and $u_i \in \Sigma_\Delta$, $\Sigma_\Delta$ is a alphabet with $\Delta$ elements. We also code each bus as $\alpha w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} \beta$, where $w_i \in \Sigma_k$, $u_i \in \Sigma_\Delta$, and $\alpha, \beta \in \Sigma_\Delta$. The length of $w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$ is equal to the length of $\alpha w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} \beta$. They both are equal to $D$. For each node $w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$, it incident to the set of hyperedges $u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D \alpha$, where $\alpha \in \Sigma_\Delta$. That is, there is an arc point to the hyperedge $u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D \alpha$ from the node $w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$. Since there are $\Delta$ elements in $\Sigma_\Delta$, each nodes incident to $\Delta$ different hyperedges. We should keep in mind that hyperedge itself is a bus. So, each node incident to $\Delta$ different buses. The node $w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$ is also incident from the set of hyperedges $\alpha w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$, where $\alpha \in \Sigma_\Delta$. That is, there is an arc point to node $w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$ from hyperedge $\alpha w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$. Since there are $\Delta$ elements in $\Sigma_\Delta$, there are $\Delta$ hyperedges connect to ( or point to ) the node $w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$. The in-degrees of each node is $\Delta$, and the out-degree of each node is $\Delta$. We can think this situation as there is a node with $\Delta$ transmitters and $\Delta$ receivers, which transmits signal to $\Delta$ buses and receives signal from $\Delta$ buses simultaneously. As to each bus, from $w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D \Rightarrow u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D \alpha$, we know that there are $k$ transmitters attached to the bus, since $w_1 \in \Sigma_k$ and $\Sigma_k$ has $k$ elements. From $\alpha w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} u_D \Rightarrow w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$, there are also $k$ receivers attached to the bus, since $w_D \in \Sigma_k$ and $\Sigma_k$ has $k$ elements.

With such an interconnected de Bruijn Network, the maximum hops needed from any node in the network to any other node ( that is the diameter of the hypergraph ) is $D$. We can simply show this as follows. In order to hop from node $w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D$ to node $\gamma_1 \lambda_1 \gamma_2 \lambda_2 \cdots \gamma_D \lambda_D$, we have

$$w_1 u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D \Rightarrow u_1 w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D \lambda_1$$

$$\Rightarrow w_2 u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D \gamma_1 \lambda_1$$

$$\Rightarrow u_2 \cdots w_{D-1} u_{D-1} w_{D} u_D \gamma_1 \lambda_1 \lambda_2$$

$$\Rightarrow \cdots$$

$$\Rightarrow \gamma_1 \lambda_1 \gamma_2 \lambda_2 \cdots \gamma_D \lambda_D.$$
The maximum number of nodes in a de Bruijn network can be calculated as follows. Each \( w_i \in \Sigma_k \) in \( w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \) takes \( k \) different values and each \( u_i \in \Sigma_\Delta \) takes \( \Delta \) different values. The total possibilities will be \((\Delta k)^D\). So, \( n=(\Delta k)^D \).

For \( UB(\Delta, D, k) \) (that is a network with \( \Delta \) T/R pairs, maximum \( D \) hops and bus volume \( k \)), we should construct the de Bruijn network \( B(\Delta/2, D, k/2) \) and replace each directed arc with undirected arc. Such the maximum number of nodes it can interconnect is \((\Delta k)^D/4\).

**Kautz bus network**

Kautz bus network \( K(\Delta, D, k) \) are constructed as follows. Similar to de Bruijn network, we code each node as an element \( w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \). It consists of \( D \) \( w_iu_i \) pairs where \( w_i \in \Sigma_k \) (\( \Sigma_k \) is an alphabet with \( k \) elements), and \( u_i \in \Sigma_\Delta \) (\( \Sigma_\Delta \) is an alphabet with \( \Delta \) elements) with the restriction that any two consecutive \( w_iu_i \) pairs are different. Although each \( w_iu_i \) pair has \( k\Delta \) options, we add a defult state ***(i.e., \( w_iu_i=***) for some i ), such, each \( w_iu_i \) has \( k\Delta+1 \) options. We also code each bus as \( \alpha w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \), where \( w_i \in \Sigma_k \), \( u_i \in \Sigma_\Delta \), and \( \alpha, \beta \in \Sigma_\Delta \). For each node \( w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \), it incident to the set of hyperedges \( u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \), according to that, 1) if \( u_D=** \), then \( \alpha \in \Sigma_\Delta \) could be any element in \( \Sigma_\Delta \), 2) if \( u_D=** \), then \( \alpha \neq u_D \), it may take any element in \( (\Sigma_\Delta-u_D) \cup (*) \). Since there are \( \Delta \) elements in \( \Sigma_\Delta \), each nodes incident to \( \Delta \) different hyperedges. The node \( w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \) is also incident from the set of hyperedges \( \alpha w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}u_D \), according to that, 1) if \( u_D=\ast \), then \( w_D=\ast \), 2) if \( u_D \in \Sigma_\Delta \), then \( w_D \in \Sigma_k \) ( \( w_D \neq \ast \)). That is, there is an arc point to node \( w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \) from hyperedge \( \alpha w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \). Since there are \( \Delta \) elements in \( \Sigma_\Delta \), there are \( \Delta \) hyperedges connect to (or point to) the node \( w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \). The in-degree of each node is \( \Delta \), and the out-degree of each node is \( \Delta \). As to each hyperedge, from \( w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \) to \( u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \alpha \), we know that there are at most \( k \) transmitters attached to the bus, since \( w_1 \in \Sigma_k \) and \( \Sigma_k \) has \( k \) elements, if \( u_1 \neq \ast \). If \( u_1=\ast \), then only one node is attached to the bus. There are also at most \( k \) receivers attached to the bus. Note that, if \( u_D=\ast \), then \( w_D=\ast \), and only one receiver is attached to the bus.

With such an interconnected Kautz Network, the maximum hops needed from any node in the network to any other node is \( D \). We can simply show this as follows. From \( w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D \) to node \( \gamma_1\lambda_1\gamma_2\lambda_2\cdots \gamma_D\lambda_D \), if \( w_Du_D \neq \gamma_1\lambda_1 \) we have

\[
\begin{align*}
    w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D &\rightarrow u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D\lambda_1 \\
    &\rightarrow w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D\gamma_1\lambda_1 \\
    &\rightarrow u_2\cdots w_{D-1}u_{D-1}w_Du_D\gamma_1\lambda_1\lambda_2 \\
    &\rightarrow \cdots \rightarrow \gamma_1\lambda_1\gamma_2\lambda_2\cdots \gamma_D\lambda_D.
\end{align*}
\]
There are at most $2D$ shifts, which visits at most $D$ hyperedges ( $D$ hops ).

If $w_Du_D = \gamma_1\lambda_1$, then $w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D = w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}\gamma_1\lambda_1$ 
$\rightarrow u_1w_2u_2\cdots w_{D-1}u_{D-1}\gamma_1\lambda_1\lambda_2 \rightarrow w_2u_2\cdots w_{D-1}u_{D-1}\gamma_1\lambda_1\gamma_2\lambda_2 \rightarrow \cdots \rightarrow \gamma_1\lambda_1\gamma_2\lambda_2\cdots \gamma_D\lambda_D$, there are at most $2D$ shifts, that is, at most $D$ hops.

The maximum number of nodes in a Kautz bus network can be calculated as follows. Each $w_iu_i \in (\Sigma_k \times \Sigma_D) \cup (\ast \ast)$ in $w_1u_1w_2u_2\cdots w_{D-1}u_{D-1}w_Du_D$ takes $\Delta k+1$ different values, and any two consecutive $w_{i-1}u_{i-1}$ and $w_iu_i$ are different. So, the first place of pair $w_iu_i$ may takes $\Delta k+1$ possibilities, and the rest each pair may takes $\Delta k$ possibilities. The total possibilities will be $(\Delta k + 1)(\Delta k)^{D-1} = (\Delta k)^D + (\Delta k)^{D-1}$. So, $n=(\Delta k)^D + (\Delta k)^{D-1}$.

For $UK(\Delta, D, k)$ (that is a network with $\Delta$ T/R pairs, maximum $D$ hops and bus volume $k$), we should construct the Kautz network $K(\Delta/2, D, k/2)$ and replace each directed arc with undirected arc. Such the maximum number of nodes it can interconnect is $(\Delta k)^D + (\Delta k)^{D-1}$.

**Bus network: dual between bus volume and T/R pairs**

Duality between bus volume $r$ and node degree $\Delta$: If bus volume $r$ and nodes degree $\Delta$ are interchangable to connect a same number of nodes, we say the network has dual property between bus volume and nodes degree. From $n = \left(\frac{\Delta r}{4}\right)^D$, we know de Bruijn network has dual property. And from $n = \left(\frac{\Delta r}{4}\right)^D + \left(\frac{\Delta r}{4}\right)^{D-1}$, Kautz network also has dual property. If network has duality property, then $n(\Delta, D, r) = n(r, D, \Delta)$. On the other hand, since $n(\Delta, D, r) = n(r, D^*, \Delta)$, we have $n(r, D, \Delta) = n(r, D^*, \Delta)$ and $D^* = D$.

In point to point communication as we discussed in section 2, one need large $\Delta$ ( large number of transmitter-receiver pairs) in order to connect larger network with the same $D$(delay). According to dual property, $\Delta$ and $r$ are interchangable, we can put buses in point to point network and reduce $\Delta$, while $D$ (delay) keep unchanged. Bus save transmitter-receiver pairs!

5. CONCLUSION

In this paper, we show what is good interconnection network and how it can be used in designing bus networks. De Bruijn network and Kautz network provide better options for engineering implemention of interconnection. For given number of transmitter-receiver pairs $\Delta$, system diameter( delay ) $D$ and the maximal number of nodes $r$ that a bus can accommodate, the total number of nodes de Bruijn network can accommodate is $\left(\frac{\Delta r}{4}\right)^D$. 
Kautz network is \((\Delta r)^D + (\Delta r)^{D-1}\). If we permit one node's transmitter and receiver on different buses, then the number of nodes de Bruijn network can accommodate is \((\Delta r)^D\).

Kautz network is \((\Delta r)^D + (\Delta r)^{D-1}\). So, the number of nodes we can connect when transmitter and receiver are separable is larger than the restricted case where the transmitter and receiver of one nodes must be on the same bus.

According to dual property, \(\Delta\) and \(r\) are interchangeable. So the significance of the bus network is, we can put buses in point to point network and reduce \(\Delta\), while \(D\) (delay) keep unchanged. Bus save transmitter-receiver pairs!

Although de Bruijn and Kautz network give a constructive way to build networks with clear parameters options, they are not optimal. There exist some networks even better than de Bruijn network or Kautz network. But due to the difficulties of constructive method, there still need a lot of effort to find out networks near Moore bound.

REFERENCES

15. D. F. Hsu(ed.) Interconnection Networks and Algorithms, Special Issue in Networks.